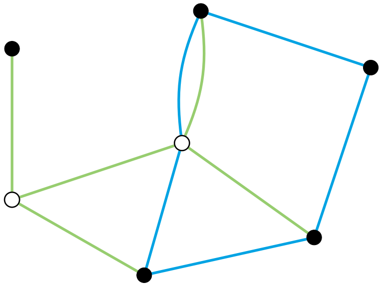
Partitioning graphs via edge-coloured homomorphisms

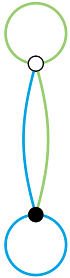
Ross Churchley

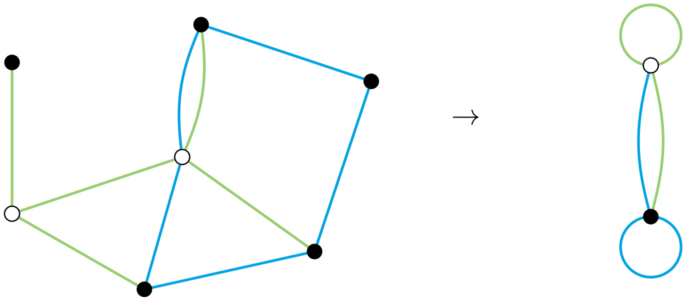
University of Victoria

1 June 2011

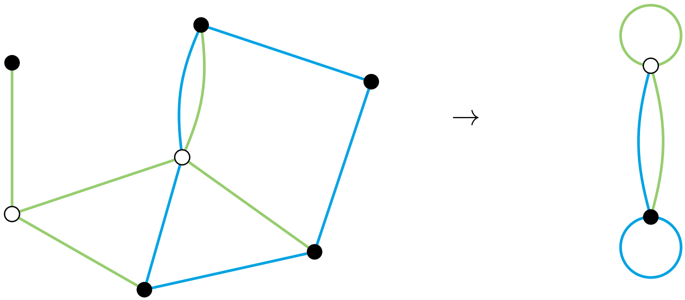


$\rightarrow$





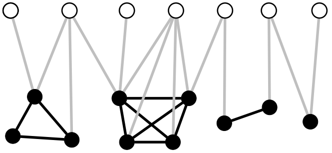
This homomorphism is a "colour-bipartition": ○ has no blue edge and • has no green edge.



"Colour-bipartitions" model 2-SAT.

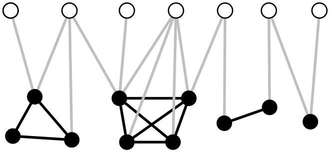
green edge uv	u, v not both $\bullet$	$(u \vee v)$
blue edge uv	u, v not both $\circ$	$(\overline{u} \vee \overline{v})$

applied to monopolar partitions



a monopolar partition divides the vertices into an independent set ○ and a disjoint union of cliques •

some small graphs:



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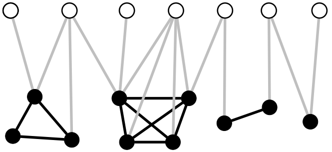

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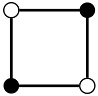

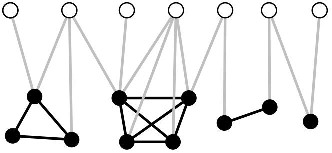

a monopolar partition divides the vertices into an independent set ∘ and a disjoint union of cliques •

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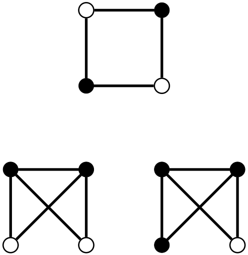

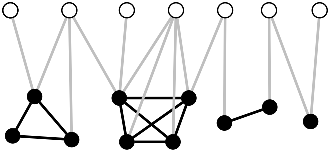

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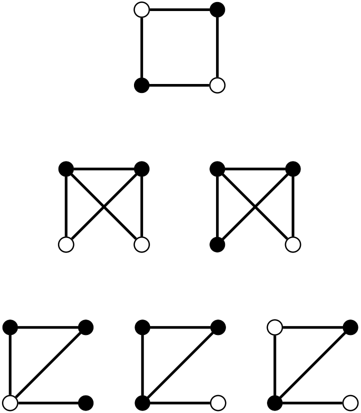

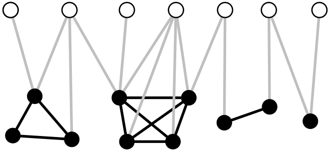

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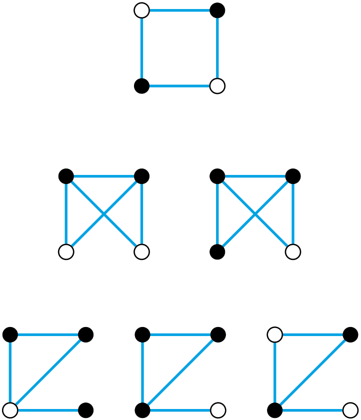

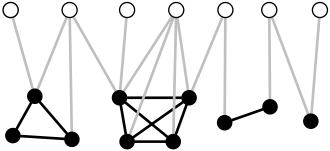

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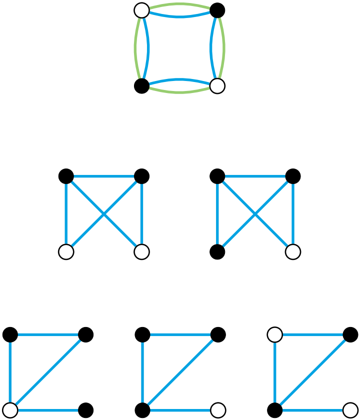

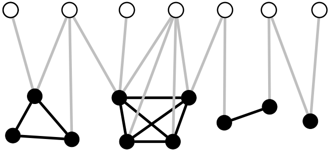

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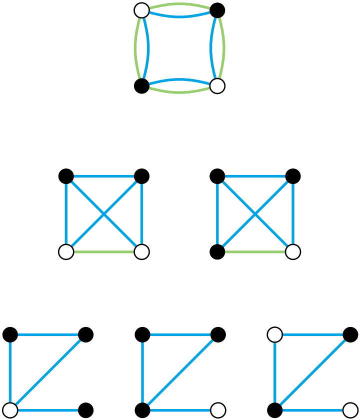

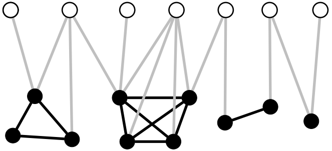

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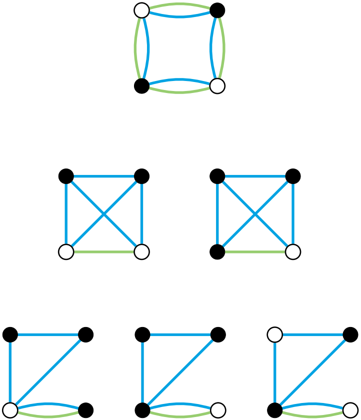
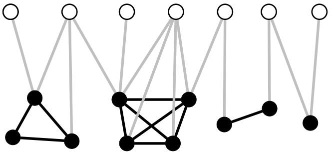
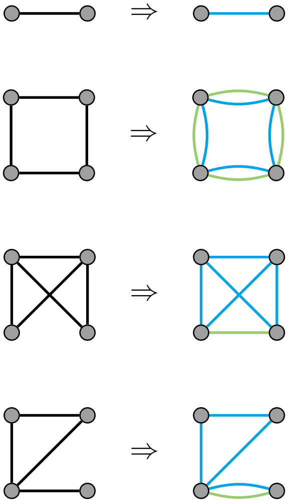
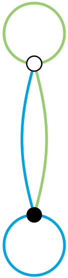


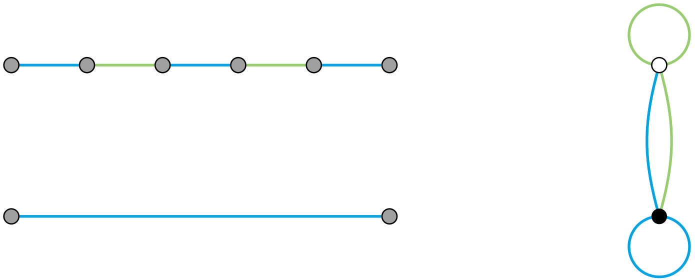




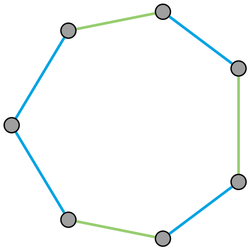
a monopolar partition divides the vertices into an independent set ∘ and a disjoint union of cliques •

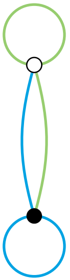
can construct H from G such that G monopolar $\Rightarrow$ H colour-bipartite useful edge-coloured structures

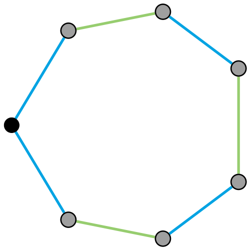


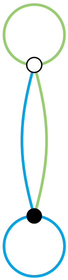


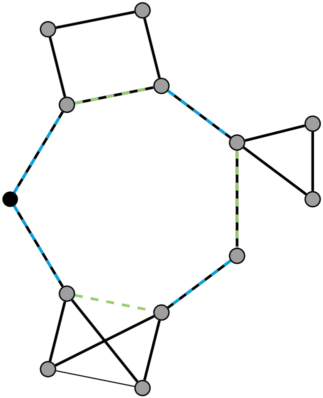
alternating walks act like edges when it comes to colour-bipartitions

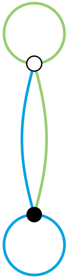


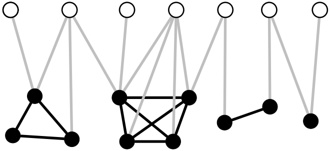


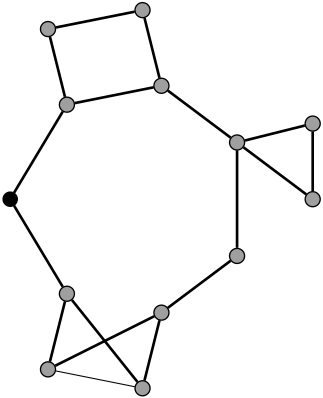


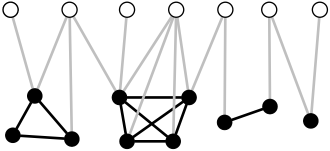






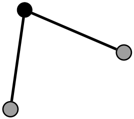


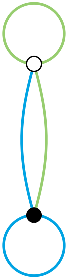


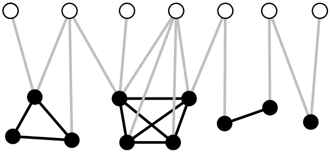


we can identify some vertices which are "forced" •

once we've found forced vertices, it has implications for other vertices







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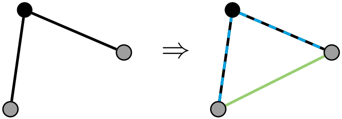

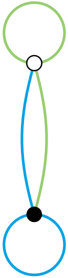

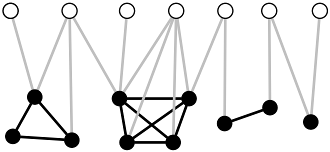

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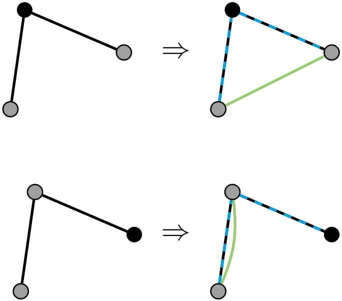

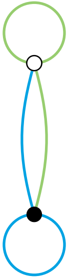

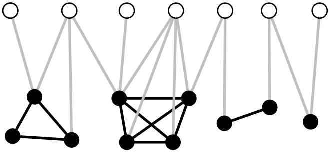

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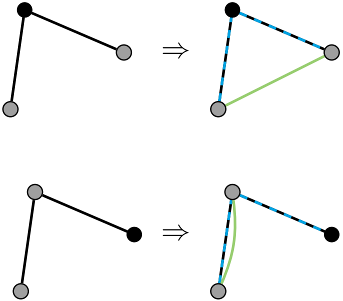

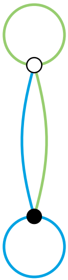

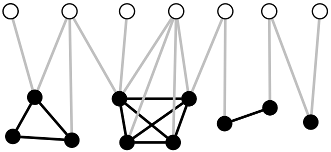

new vertices are forced; repeat

recap

1. Input graph G.

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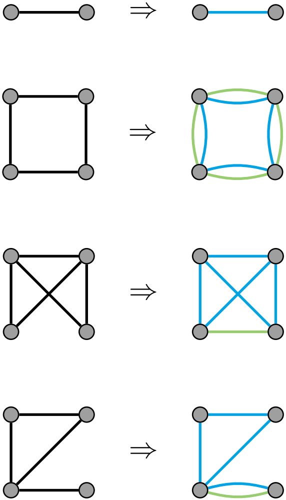
2. Construct edge-coloured H according to induced subgraphs of G



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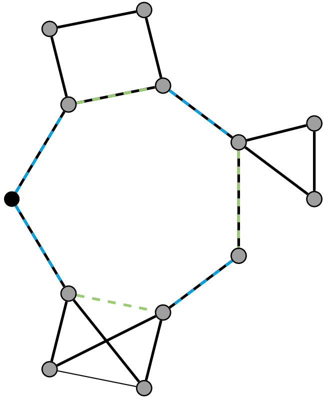
2. Construct edge-coloured H according to induced subgraphs of G
3. Find “forced” vertices



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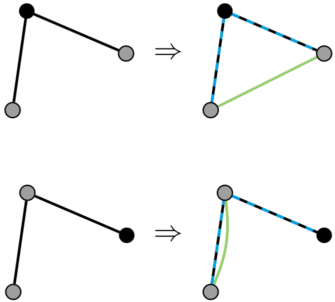
2. Construct edge-coloured H according to induced subgraphs of G
3. Find "forced" vertices
4. Add green edges to H



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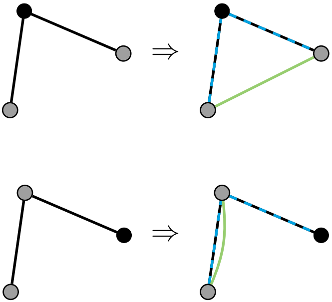
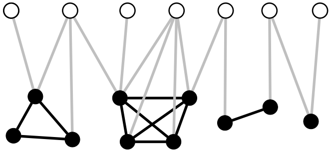
3. Find "forced" vertices
4. Add green edges to H
5. Repeat 3 and 4 until no new forced vertices



1. Input graph G.
2. Construct edge-coloured H according to induced subgraphs of G
3. Find "forced" vertices
4. Add green edges to H
5. Repeat 3 and 4 until no new forced vertices
6. If G is monopolar, then H is colour-bipartite.



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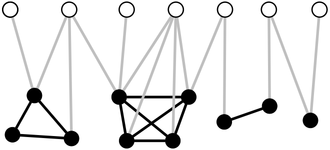

Question: when is the converse true?

Theorem

Suppose G is claw-free. Then G is monopolar if and only if H is colour-bipartite.

Theorem

Suppose G is claw-free. Then G is monopolar if and only if H is colour-bipartite. In fact, for each (unforced) v there is a monopolar partition with v in the $\circ$ independent set.

Theorem

Suppose G **is such that every induced** P_3 **satisfies at least one of the following:**

- **is centred on a vertex of degree 2, or**
- **shares a vertex with some triangle of** G**, or**
- **shares an edge with some induced** C_4 **of** G**, or**
- **is not contained in any induced cycle of length** ≥ 5**.**

Then G is monopolar if and only if H is colour-bipartite. In fact, for each (unforced) v there is a monopolar partition with v in the $\circ$ independent set.

Why this class?

Suppose G is such that every induced P_3 satisfies at least one of the following:

- is centred on a vertex of degree 2, or
- shares a vertex with some triangle of G, or
- shares an edge with some induced C_4 of G, or
- is not contained in any induced cycle of length ≥ 5.

Includes...

- claw-free graphs
- chordal graphs
- permutation graphs
- co-comparability graphs
- graphs which contain no induced cycle of length ≥ 5

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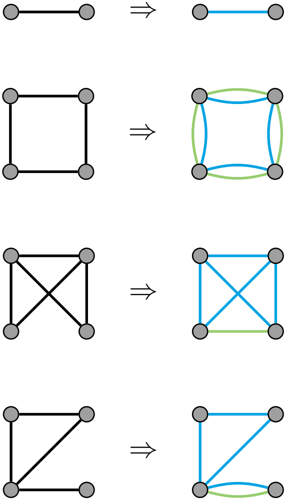

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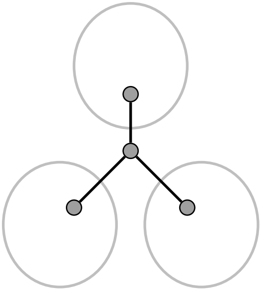

If the P_3 has no green edge, no long cycle means the graph breaks into components we can deal with.

Theorem

Suppose G contains no P_3 which

- is centred on a vertex of degree ≥ 3,
- is vertex-disjoint from every triangle of G,
- is edge-disjoint from every C_4 of G, and
- is contained in a cycle of length ≥ 5.

Then G is monopolar if and only if H is colour-bipartite. In fact, for each (unforced) v there is a monopolar partition with v in the $\circ$ independent set.

Proof

By induction (on the number of vertices of degree ≥ 3 which are disjoint from all triangles). □

- monopolar partitions
  - claw-free, chordal, permutation, co-comparability...

- bipartitions
- split partitions
- monopolar partitions
  - claw-free, chordal, permutation, co-comparability...

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- split partitions
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- partition into a clique and disjoint union of cliques

- bipartitions
- split partitions
- monopolar partitions
  - claw-free, chordal, permutation, co-comparability...
- partition into a clique and disjoint union of cliques
- partition into a clique and a triangle-free graph

- bipartitions
- split partitions
- monopolar partitions
  - claw-free, chordal, permutation, co-comparability...
- partition into a clique and disjoint union of cliques
- partition into a clique and a triangle-free graph
- partition into an independent set and a triangle-free graph
  - restricted classes, e.g. paw-free

Partitioning graphs via edge-coloured homomorphisms

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University of Victoria

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