Monopolar claw-free graphs

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On the social dynamics of small children

It had been a long year, thought Mrs Bradshaw, but it was almost over. She looked out the window of her second-grade classroom as her students filed in. Sunlight glistened invitingly off the playground slide. "Can we go outside and play?" asked Emily. It wasn't a bad idea. Her class certainly wouldn't be getting any work done in the hour that separated them from summer holidays. But...

"I get to play with Madison!" "No, I do!" Mrs Bradshaw sighed. "We'll see, Emily. For now, find your desks and wait quietly."

"I want to play dodgeball!!!!" shouted Connor. Gustav, who had been sent to the medical office last time, whimpered. Dodgeball was trickier. Her students learned the word 'alliance' last month, and had quickly discovered the related idea of ganging up on somebody.

"What do you think?" Mrs Bradshaw asked Ms Cook, her student teacher. "Could we do both? I can organize dodgeball outside if you supervise the rest. If we choose the groups right, we might make it to 2:30 without a problem..."

"Sounds good," replied Ms Cook, "but how are we going to find the right groups?"

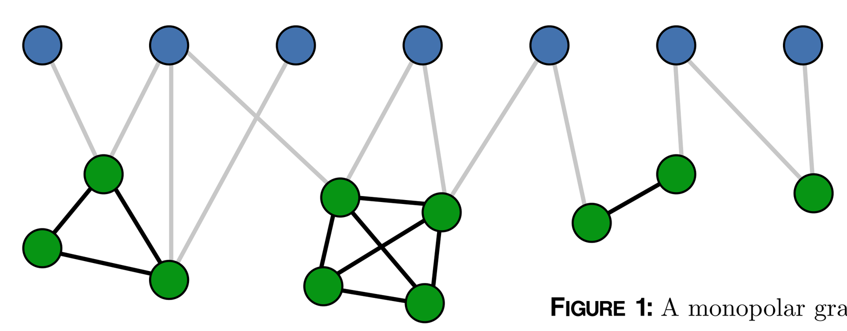

FIGURE 1: A monopolar graph

War and cliques

The solution to Mrs Bradshaw's predicament lies in the diagrams studied in graph theory. A graph $G=(V,E)$ is a set V of vertices and a set E of edges which connect two vertices each. We can model the classroom by drawing a vertex for each student and connecting friends with edges. To make it to the end of the day without a conflict, Mrs Bradshaw and Ms Cook have to split up the vertices into a blue dodgeball group and a green playground group in a particular way: no two blue vertices should be friends (and potential allies), and the green vertices should be clumped into "cliques" of friends who can all play nicely together. In graph-theoretic terms, we must find a *monopolar partition* of the classroom.

Not every graph is *monopolar*; in other words, it might not be possible to find such a partition of the classroom. Determining whether a graph is monopolar — and finding a monopolar partition if one exists — is a hard problem in general. Fortunately, there is an efficient solution if the particular graph is claw free. By happy coincidence, Mrs Bradshaw happens to know that no student in her class is friends with three mutual enemies, and she will be able to use our results.

Feathers and sequence

The key to finding monopolar partitions of claw-free graphs is a structure called the *well-formed sequence*. A sequence of vertices $v_1, v_2, v_3, \ldots, v_k$ is said to be well-formed in the graph G if the following two conditions are true:

- for every even index i, the vertices v_{i-1} and v_i are adjacent;
- for every odd index i, G contains adjacent vertices w, x such that if v_{i-1} and v_i are adjacent, w and x are both connected to v_{i-1} but not v_i, or vice versa; and if v_{i-1} and v_i are nonadjacent, w and x are each connected to v_{i-1} and v_i. For the sake of brevity, we will call such a pair an *odd pair*.

A picture is often better than words for definitions like these, so an example of a well-formed sequence is given above. For the index $i=5$, the vertices v_4 and v_5 are adjacent, so v_3 and the grey vertex are both connected to v_4 but not v_5. The opposite happens for the index $i=7$: v_8 and the grey vertex are both connected to v_7 but not v_6. Finally, since v_{10} and v_{11} are nonadjacent we want there to be two adjacent vertices connected to both v_{10} and v_{11}, as the grey vertices above them are.

Well-formed sequences are important because they force the colour of certain vertices: if the first vertex in a well-formed sequence is coloured blue, then the sequence has to alternate blue, green, blue, green, blue, green as above. It is not too difficult, after seeing some examples, to see why this is the case.

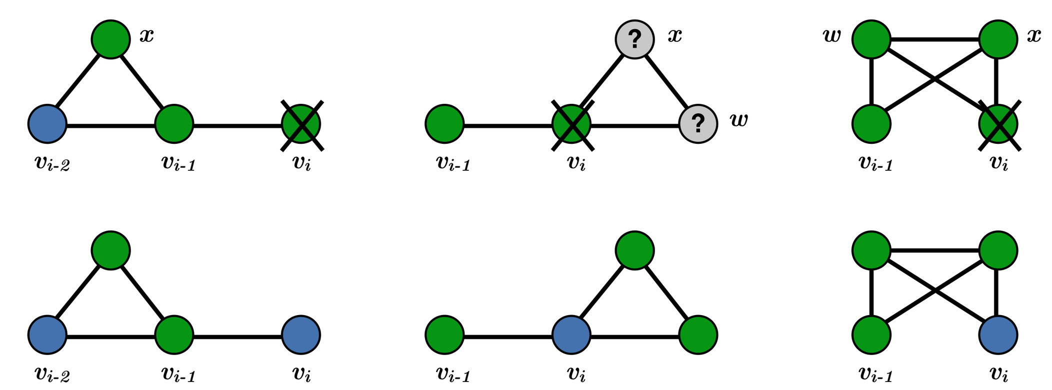

FIGURE 2: The reason why a well-formed sequence beginning with a blue vertex must alternate in colour. Since we don't want blue friends, a blue vertex must be followed by a green one. These three cases show why a green even-indexed vertex v_{i-1} must be followed by a blue vertex v_i. In the first case, v_i cannot be green because x, v_{i-1}, v_i do not break nicely into cliques. In the second case, w and x would both have to be blue if v_i were green, which is a problem since we don't want blue friends. In the third case, w and x must both be green, so v_i must be blue for the green vertices to split into cliques.

What happens if a well-formed sequence contains the same vertex twice? If a well-formed sequence starts and ends with the same vertex, it can't possibly alternate colours if it has an even length! If a vertex is in this situation, we say that it is *self-incompatible*, and must colour it green. As it turns out, we can tell if a claw-free graph is monopolar just by looking at these special vertices!

Theorem 1

A claw-free graph is monopolar if and only if the set of self-incompatible vertices can be broken into (disjoint) cliques.

In which Mrs Bradshaw uses graph theory

"All right," announced Mrs Bradshaw, "we're going to line up to go outside. Connor, you're our special helper today, so you can take a friend and line up first." Better let Connor play dodgeball, she thought, or he'll throw a fit. And that means Eric and Joel, his partners in crime, should go to the playground. Now let's see... "Sarah, you can line up." Sarah was friends with Eric but always fought with Joel, so she'd have to play dodgeball. "Madison, you can line up." She was friends with Sarah. "Jacob, Rebecca.... June, Alex, Brandon..." Mrs Bradshaw called her students' names one by one in a well-formed sequence. She could only hope that forming the right groups in her class was possible at all...

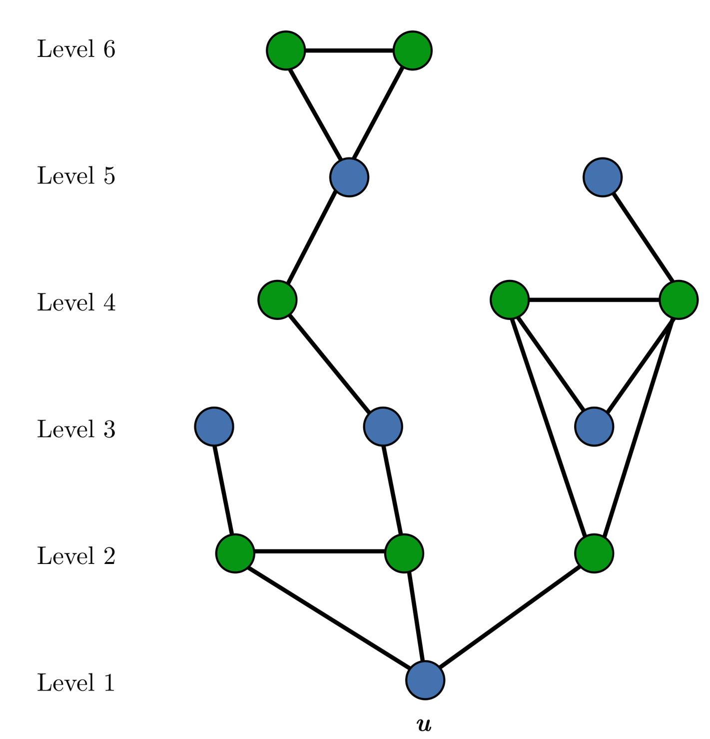

FIGURE 3: A typical "tree," with root vertex u, found by our algorithm. In order to produce this tree, we consider the vertices of each each level in turn. If the current level is odd, we add to the next level the friends of the vertices in this level. On the other hand, if the current level is even, then we consider each vertex v in the current level and add every vertex w such that v, w are an odd pair. We continue this process until there are no more vertices to process.
We can determine whether u is self-incompatible by checking pairs of vertices in this tree. If any two vertices in odd levels are adjacent, or if any two vertices in even levels form an odd pair, then u is self-incompatible and we must colour it green. Otherwise, u is not self-incompatible and we can colour the entire tree: blue for vertices in odd levels, and green for those in even levels.

Algorithm and blues

The characterization in Theorem 1 gives a simple and efficient algorithm for finding monopolar partitions in claw-free graphs. Although this might be a little technical for Mrs Bradshaw's purposes, it is of interest to computer scientists and other experts.

The algorithm is a modified version of the well-known breadth-first search algorithm. Starting with a root vertex u, we create a "tree" whose branches are well-formed sequences in G. By examining this tree, it is possible to determine whether u is self-incompatible. If it is, we know that u must be coloured green; otherwise, we colour u blue and alternate colours going up the tree. If necessary, we repeat this with another vertex until all the vertices are coloured.

Theorem 2

There is an $O(n^3)$ algorithm to determine whether a claw-free graph is monopolar, and to find a monopolar partition if one exists.

A happy ending to the school year

Mrs Bradshaw led her students outside. "Do you think this is going to work?" whispered Ms Cook. "I sure hope so," replied Mrs Bradshaw, "I'm pretty sure it will, but you never know with this class."

An hour later, the students were back inside, cleaning out their cubbies and getting ready to go home. Their eyes were bright with excitement for the holidays. One by one, Mrs Bradshaw's students came up to say goodbye. "Thanks for letting us play dodgeball!" "I'm glad we got to play with our friends on the last day of school." Even Gustav — the only casualty of the day with a scraped knee — managed a tearful smile as he hugged his teacher.

Next year, Mrs Bradshaw thought, she'd miss the chaos of this class.

Polar claw-free graphs

Monopolar partitions are usually studied as a special case of *polar partitions*. If Mrs Bradshaw's classroom hadn't been monopolar, perhaps she would have decided that two dodgeball-playing friends would not cause a problem if they had no common enemy. In this case, she would be looking for a polar partition of her classroom. This turns out to be a difficult problem in general, even if the graph is claw-free.

Theorem 3

The problem of recognizing polar claw-free graphs is NP-complete.

Further reading

Our results on monopolar claw-free graphs follow earlier ones on line graphs.

R. Churchley and J. Huang, The polarity and monopolarity of claw-free graphs, manuscript (2010)

R. Churchley and J. Huang, List monopolar partitions of claw-free graphs, manuscript (2010)

R. Churchley and J. Huang, Line-polar graphs: characterization and recognition, submitted (2009)

T. Ekim and J. Huang, Recognizing line-polar bipartite graphs in time O(n), manuscript (2009)

J. Huang and B. Xu, A forbidden subgraph characterization of line-polar bipartite graphs, Discrete Applied Math. 158 (2010) 666-680.

T. Ekim, Polarity of claw-free graphs, manuscript (2009)

Although the problem of recognizing polar claw-free graphs in NP-complete, efficient algorithms have been found for recognizing other classes of polar graphs.

T. Ekim, P. Heggernes, and D. Meister, Polar permutation graphs, J. Fiala, J. Kratochvil, and M. Miller (eds.): IWOCA 2009, LNCS 5874, pp 218-229, 2009.

T. Ekim, N. V. R. Mahadev, and D. de Werra, Polar cographs, Discrete Applied Math. 156 (2008) 1652-1660.

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Z.A. Chernyak and A. A. Chernyak, About recognizing (alpha, beta)-classes of polar graphs, Discrete Math. 62 (1986) 133-138.